Convex Sets and Minimal Sublinear Functions

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Abstract

We show that, given a closed convex set K containing the origin in its interior, the support function of the set $\{y \in K^* \mid \exists x \in K \text{ such that } \langle x, y\rangle = 1\}$ is the pointwise smallest among all sublinear functions σ such that $K = \{x \mid \sigma(x) \leq 1\}$.

1 Introduction

The purpose of this note is to prove the following theorem. For $K \subseteq \mathbb{R}^n$, we use the notation

$$\begin{aligned} K^* &= \{y \in \mathbb{R}^n \mid \langle x, y\rangle \leq 1 \text{ for all } x \in K\} \\ \hat{K} &= \{y \in K^* \mid \langle x, y\rangle = 1 \text{ for some } x \in K\}. \end{aligned}$$

The set K^* is the *polar* of K. The set $\hat{K}$ is contained in the relative boundary of K^*. The polar K^* is a convex set whereas $\hat{K}$ is not convex in general.

The *support function* of a nonempty set $T \subset \mathbb{R}^n$ is defined by

$$\sigma_T(x) = \sup_{y \in T} \langle x, y\rangle \quad \text{for all } x \in \mathbb{R}^n.$$

It is straightforward to show that support functions are *sublinear*, that is they are convex and positively homogeneous (A function $f : \mathbb{R}^n \to \mathbb{R}$ is *positively homogeneous* if $f(tx) = tf(x)$ for every $x \in \mathbb{R}^n$ and $t > 0$), and $\sigma_T = \sigma_{\overline{\text{conv}}(T)}$ [5]. We will show that, if $K \subset \mathbb{R}^n$ is a closed convex set containing the origin in its interior, then $K = \{x \mid \sigma_{\hat{K}}(x) \leq 1\}$. The next theorem shows that $\sigma_{\hat{K}}$ is the smallest sublinear function with this property.

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14. ABSTRACT
We show that, given a closed convex set K containing the origin in its interior the support function of the set fy 2 K? j 9x 2 K such that hx; yi = 1g is the pointwise smallest among all sublinear functions ? such that K = fx j ?(x) ? 1g.

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Theorem 1 *Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. If $\sigma : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function such that $K = \{x \in \mathbb{R}^n \,|\, \sigma(x) \leq 1\}$, then $\sigma_{\hat{K}}(x) \leq \sigma(x)$ for all $x \in \mathbb{R}^n$.*

In the remainder we define $\rho_K = \sigma_{\hat{K}}$.

Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. A standard concept in convex analysis [5, 7] is that of *gauge* (sometimes called Minkowski function), which is the function γ_K defined by

$$\gamma_K(x) = \inf\{t > 0 \,|\, t^{-1}x \in K\} \quad \text{for all } x \in \mathbb{R}^n.$$

By definition γ_K is nonnegative. One can readily verify that $K = \{x \,|\, \gamma_K(x) \leq 1\}$. It is well known that γ_K is the support function of K^* (see [5] Proposition 3.2.4).

Given any sublinear function σ such that $K = \{x \,|\, \sigma(x) \leq 1\}$, it follows from positive homogeneity that $\sigma(x) = \gamma_K(x)$ for every x where $\sigma(x) > 0$. Hence $\sigma(x) \leq \gamma_K(x)$ for all $x \in \mathbb{R}^n$. On the other hand, we prove in Theorem 1 that the sublinear function ρ_K satisfies $\rho_K(x) \leq \sigma(x)$ for all $x \in \mathbb{R}^n$. In words, γ_K is the largest sublinear function such that $K = \{x \in \mathbb{R}^n \,|\, \sigma(x) \leq 1\}$ and ρ_K is the smallest.

Note that ρ_K can take negative values, so in general it is different from the gauge γ_K. Indeed the recession cone of K, which is the set $\text{rec}(K) = \{x \in K \,|\, tx \in K \text{ for all } t > 0\}$, coincides with $\{x \in K \,|\, \sigma(x) \leq 0\}$ for every sublinear function σ such that $K = \{x \,|\, \sigma(x) \leq 1\}$. In particular $\rho_K(x)$ can be negative for $x \in \text{rec}(K)$. For example, let $K = \{x \in \mathbb{R}^2 \,|\, x_1 \leq 1,\, x_2 \leq 1\}$. Then $K^* = \text{conv}\{(0,0),(1,0),(0,1)\}$ and $\hat{K} = \text{conv}\{(1,0),(0,1)\}$. Therefore, for every $x \in \mathbb{R}^2$, $\gamma_K(x) = \max\{0, x_1, x_2\}$ and $\rho_K(x) = \max\{x_1, x_2\}$. In particular, $\rho_K(x) < 0$ for every x such that $x_1 < 0$, $x_2 < 0$.

By Hörmander's theorem [6], a sublinear function $\sigma : \mathbb{R}^n \to \mathbb{R}$ is the support function of a unique bounded closed convex set $C \subset \mathbb{R}^n$, say $\sigma = \sigma_C$. So the condition $K = \{x \in \mathbb{R}^n \,|\, \sigma(x) \leq 1\}$ says $K = C^*$. Thus Theorem 1 can be restated in its set version.

Theorem 2 *Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. If $C \subset \mathbb{R}^n$ is a bounded closed convex set such that $K = C^*$, then $\hat{K} \subset C$.*

When K is bounded, this theorem is trivial (the hypothesis $K = C^*$ becomes $K^* = C$. The conclusion $\hat{K} \subset C$ is obvious because one always has $\hat{K} \subset K^*$.) So the interesting case of Theorem 1 is when K is unbounded.

We present the proof of Theorem 1 in Section 2. Theorem 1 has applications in integer programming. In particular it is used to establish the relationship between minimal inequalities and maximal lattice-free convex sets [1], [2]. We summarize these results in Section 3.

2 Proof of Theorem 1

We will need Straszewicz's theorem [8] (see [7] Theorem 18.6). Given a closed convex set C, a point $x \in C$ is *extreme* if it cannot be written as a proper convex combination of two distinct points in C. A point $x \in C$ is *exposed* if there exists a supporting hyperplane H for C such that $H \cap C = \{x\}$. Clearly exposed points are extreme. We will denote by $\mathrm{ext}(C)$ the set of extreme points and $\exp(C)$ the set of exposed points of C.

Theorem 3 *Given a closed convex set C, the set of exposed points of C is a dense subset of the set of extreme points of C.*

Let K be a closed convex set containing the origin in its interior. Let σ be a sublinear function such that $K = \{x \,|\, \sigma(x) \leq 1\}$. The boundary of K, denoted by $\mathbf{bd}(K)$, is the set $\{x \in K \,|\, \sigma(x) = 1\}$.

Lemma 4 *For every $x \notin \mathrm{rec}(K)$, $\sigma(x) = \rho_K(x) = \sup_{y \in K^*} \langle x, y \rangle$.*

Proof. Let $x \notin \mathrm{rec}(K)$. Then $t = \sigma(x) > 0$. By positive homogeneity, $\sigma(t^{-1}x) = 1$, hence $t^{-1}x \in \mathbf{bd}(K)$. Since K is closed and convex, there exists a supporting hyperplane for K containing $t^{-1}x$. Since $0 \in \mathbf{int}(K)$, this implies that there exists $\bar{y} \in K^*$ such that $(t^{-1}x)\bar{y} = 1$. In particular $\bar{y} \in \hat{K}$, hence by definition $\rho_K(x) \geq \langle x, \bar{y} \rangle = t$.

Furthermore, for any $y \in K^*$, $\langle t^{-1}x, y \rangle \leq 1$, hence $\langle x, y \rangle \leq t$, which implies $t \geq \sup_{y \in K^*} \langle x, y \rangle$. Thus

$$\rho_K(x) \geq t \geq \sup_{y \in K^*} \langle x, y \rangle \geq \sup_{y \in \hat{K}} \langle x, y \rangle = \rho_K(x),$$

where the last inequality holds since $\hat{K} \subset K^*$, hence equality holds throughout. $\square$

Corollary 5 $K = \{x \,|\, \rho_K(x) \leq 1\}$.

Lemma 6 *Given an exposed point $\bar{y}$ of K^* different from the origin, there exists $x \in K$ such that $\langle x, \bar{y} \rangle = 1$ and $\langle x, y \rangle < 1$ for all $y \in K^*$ distinct from $\bar{y}$.*

Proof. If $\bar{y} \neq 0$ is an exposed point of K^*, then there exists a supporting hyperplane $H = \{y \,|\, \langle a, y \rangle = \beta\}$ such that $\langle a, \bar{y} \rangle = \beta$ and $\langle a, y \rangle < \beta$ for every $y \in K^* \setminus \{\bar{y}\}$. Since $0 \in K^*$ and $\bar{y} \neq 0$, $\beta > 0$. Thus the point $x = \beta^{-1}a \in K^{**} = K$ satisfies the statement (where $K = K^{**}$ holds because $0 \in K$). $\square$

The next lemma states that $\hat{K}$ and $\hat{K} \cap \exp(K^*)$ have the same support function.

Lemma 7 *For every $x \in \mathbb{R}^n$, $\rho_K(x) = \sup_{y \in \hat{K} \cap \exp(K^*)} \langle x, y \rangle$.*

Proof. We first show that $\rho_K(x) = \sup_{y \in \hat{K} \cap \mathrm{ext}(K^*)} \langle x, y \rangle$. Given $y \in \hat{K}$ we show that there exists an extreme point y' of K^* in $\hat{K}$ such that $\langle x, y \rangle \leq \langle x, y' \rangle$. Since $y \in \hat{K}$, there exists $\bar{x} \in K$ such that $\langle \bar{x}, y \rangle = 1$. The point y is a convex combination of extreme points $y_1, \dots, y_k$ of K^*, and each y_i satisfies $\langle \bar{x}, y^i \rangle = 1$. Thus $y^1, \dots, y^k \in \hat{K}$, and $\langle x, y^i \rangle \geq \langle x, y \rangle$ for at least one i.

By Straszewicz's theorem (Theorem 3) the set of exposed points in K^* is a dense subset of the extreme points of K^*. By Lemma 6, all exposed points of K^* except the origin are in $\hat{K}$, hence $\exp(K^*) \cap \hat{K}$ is dense in $\mathrm{ext}(K^*) \cap \hat{K}$. Therefore $\rho_K(x) = \sup_{y \in \hat{K} \cap \exp(K^*)} \langle x, y \rangle$. ☐

A function σ is *subadditive* if $\sigma(x_1 + x_2) \leq \sigma(x_1) + \sigma(x_2)$ for every $x_1, x_2 \in \mathbb{R}^n$. It is easy to show that σ is sublinear if and only if it is subadditive and positively homogeneous.

Proof of Theorem 1. By Lemma 4, we only need to show $\sigma(x) \geq \rho_K(x)$ for points $x \in \mathrm{rec}(K)$. By Lemma 7 it is sufficient to show that, for every exposed point $\bar{y}$ of K^* contained in $\hat{K}$, $\sigma(x) \geq \langle x, \bar{y} \rangle$.

Let $\bar{y}$ be an exposed point of K^* in $\hat{K}$. By Lemma 6 there exists $\bar{x} \in K$ such that $\langle \bar{x}, \bar{y} \rangle = 1$ and $\langle \bar{x}, y \rangle < 1$ for all $y \in K^*$ distinct from $\bar{y}$. Note that $\bar{x} \in \mathbf{bd}(K)$.

We observe that for all $\delta > 0$, $\bar{x} - \delta^{-1} x \notin \mathrm{rec}(K)$. Indeed, since $x \in \mathrm{rec}(K)$, $\bar{x} + \delta^{-1} x \in K$. Hence $\bar{x} - \delta^{-1} x \notin \mathbf{int}(K)$ because $\bar{x} \in \mathbf{bd}(K)$. Since $0 \in \mathbf{int}(K)$ and $\bar{x} - \delta^{-1} x \notin \mathbf{int}(K)$, then $\bar{x} - \delta^{-1} x \notin \mathrm{rec}(K)$. Thus by Lemma 4

$$\sigma(\bar{x} - \delta^{-1} x) = \sup_{y \in K^*} \langle \bar{x} - \delta^{-1} x, y \rangle. \tag{1}$$

Since $\bar{x} \in \mathbf{bd}(K)$, $\sigma(\bar{x}) = 1$. By subadditivity, $1 = \sigma(\bar{x}) \leq \sigma(\bar{x} - \delta^{-1} x) + \sigma(\delta^{-1} x)$. By positive homogeneity, the latter implies that $\sigma(x) \geq \delta - \delta \sigma(\bar{x} - \delta^{-1} x)$ for all $\delta > 0$. By (1),

$$\sigma(x) \geq \inf_{y \in K^*} [\delta(1 - \langle \bar{x}, y \rangle) + \langle x, y \rangle], \quad \forall \delta > 0$$

hence

$$\sigma(x) \geq \sup_{\delta > 0} \inf_{y \in K^*} [\delta(1 - \langle \bar{x}, y \rangle) + \langle x, y \rangle].$$

Let $g(\delta) = \inf_{y \in K^*} \delta(1 - \langle \bar{x}, y \rangle) + \langle x, y \rangle$. Since $\bar{x} \in K$, $1 - \langle \bar{x}, y \rangle \geq 0$ for every $y \in K^*$. Hence $\delta(1 - \langle \bar{x}, y \rangle) + \langle x, y \rangle$ defines an increasing affine function of δ for each $y \in K^*$, therefore $g(\delta)$ is increasing and concave. Thus $\sup_{\delta > 0} g(\delta) = \lim_{\delta \to \infty} g(\delta)$.

Since $0 \in \mathbf{int}(K)$, K^* is compact. Hence, for every $\delta > 0$ there exists $y(\delta) \in K^*$ such that $g(\delta) = \delta(1 - \langle \bar{x}, y(\delta) \rangle) + \langle x, y(\delta) \rangle$. Furthermore, there exists a sequence $(\delta_i)_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty} \delta_i = +\infty$ and the sequence $(y_i)_{i \in \mathbb{N}}$ defined by $y_i = y(\delta_i)$ converges, because in a compact set every sequence has a convergent subsequence. Let $y^* = \lim_{i \to \infty} y_i$.

We conclude the proof by showing that $\sigma(x) \geq \langle x, y^* \rangle$ and $y^* = \bar{y}$.

$$\begin{aligned}
\sigma(x) \geq \sup_{\delta>0} g(\delta) &= \lim_{i\to\infty} g(\delta_i) \\
&= \lim_{i\to\infty} [\delta_i(1 - \langle \bar{x}, y_i \rangle) + \langle x, y_i \rangle] \\
&= \lim_{i\to\infty} \delta_i(1 - \langle \bar{x}, y_i \rangle) + \langle x, y^* \rangle \\
&\geq \langle x, y^* \rangle
\end{aligned}$$

where the last inequality follows from the fact that $\delta_i(1 - \langle \bar{x}, y_i \rangle) \geq 0$ for all $i \in \mathbb{N}$. Finally, since $\lim_{i\to\infty} \delta_i(1 - \langle \bar{x}, y_i \rangle)$ is bounded and $\lim_{i\to\infty} \delta_i = +\infty$, it follows that $\lim_{i\to\infty}(1 - \langle \bar{x}, y_i \rangle) = 0$, hence $\langle \bar{x}, y^* \rangle = 1$. By our choice of $\bar{x}$, $\langle \bar{x}, y \rangle < 1$ for every $y \in K^*$ distinct from $\bar{y}$. Hence $y^* = \bar{y}$. □

3 An application to integer programming

In [1] and [2], Basu et al. apply Theorem 1 to cutting plane theory.

Consider a mixed integer linear program, and the optimal tableau of the linear programming relaxation. We select n rows of the tableau, relative to n basic integer variables $x_1, \dots, x_n$. Let $s_1, \dots, s_m$ denote the nonbasic variables. Let f_i be the value of x_i in the basic solution associated with the tableau, $i = 1, \dots, n$, and suppose $f \notin \mathbb{Z}^n$. The tableau restricted to these n rows is of the form

$$x = f + \sum_{j=1}^{m} r^j s_j, \quad x \in P \cap \mathbb{Z}^n,\ s \geq 0,\ \text{ and } s_j \in \mathbb{Z}, j \in I, \tag{2}$$

where $r^j \in \mathbb{R}^n$, $j = 1, \dots, m$, I denotes the set of integer nonbasic variables, and P is some full-dimensional rational polyhedron in $\mathbb{R}^n$, representing constraints on the basic variables (typically nonnegativity or bounds on the variables).

An important question in integer programming is to derive valid inequalities for (2), cutting off the current infeasible solution $x = f$, $s = 0$. We consider a simplified model where the integrality conditions are relaxed on all nonbasic variables. So we study the following model, introduced by Johnson [4],

$$x = f + \sum_{j=1}^{m} r^j s_j, \quad x \in S,\ s \geq 0, \tag{3}$$

where where $S = P \cap \mathbb{Z}^n$ and $f \in \mathrm{conv}(S) \setminus \mathbb{Z}^n$. Note that every inequality cutting off the point $(f, 0)$ can be expressed in terms of the nonbasic variables s only, and can therefore be written in the form $\sum_{j=1}^{m} \alpha_j s_j \geq 1$.

Basu et al. [2] study "general formulas" to generate such inequalities. By this, we mean functions $\psi : \mathbb{R}^n \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{m} \psi(r^j)s_j \geq 1$$

is valid for (3) for every choice of m and vectors $r^1, \ldots, r^m \in \mathbb{R}^n$. We refer to such functions ψ as *valid functions* (with respect to f and S). Since one is interested in the deepest inequalities cutting off $(f, 0)$, one only needs to investigate (pointwise) *minimal valid functions*.

Given a sublinear function ψ such that the set

$$B_\psi = \{x \in \mathbb{R}^n \,|\, \psi(x - f) \leq 1\} \tag{4}$$

is *S-free* (i.e. $\mathbf{int}(B_\psi) \cap S = \emptyset$), it is easily shown that ψ is a valid function. Indeed, since ψ is sublinear, B_ψ is a closed convex set with f in its interior, thus, given any solution $(\bar{x}, \bar{s})$ to (3), we have $\sum_{j=1}^{m} \psi(r^j)\bar{s}_j \geq \psi(\sum_{j=1}^{m} r^j \bar{s}_j) = \psi(\bar{x} - f) \geq 1$, where the first inequality follows from sublinearity and the last one from the fact that $\bar{x} \notin \mathbf{int}(B_\psi)$.

On the other hand, Dey and Wolsey [3] show that, if ψ is a minimal valid function, ψ is sublinear and B_ψ is an S-free convex set with f in its interior.

Using Theorem 1, Basu et al. [2] prove that, if ψ is a minimal valid function, then B_ψ is a *maximal S-free convex set*. That is, B_ψ is an inclusionwise maximal convex set such that $\mathbf{int}(B_\psi) \cap S = \emptyset$. Furthermore, they give a explicit formulas for all minimal valid functions.

In order to prove this result, they first show that maximal S-free convex sets are polyhedra. Therefore, a maximal S-free convex set $B \subseteq \mathbb{R}^n$ containing f in its interior can be uniquely written in the form $B = \{x \in \mathbb{R}^n : \langle a_i, x - f\rangle \leq 1,\ i = 1, \ldots, k\}$. Thus, if we let $K := \{x - f \,|\, x \in B\}$, Theorem 1 implies that the function $\psi_B := \rho_K$ is the minimal sublinear function such that $B = \{x \in \mathbb{R}^n \,|\, \psi_B(x - f) \leq 1\}$. Note that, since $K^* = \mathrm{conv}\{0, a_1, \ldots, a_k\}$, ψ_B has the following simple form

$$\psi_B(r) = \max_{i=1,\ldots,k} \langle a_i, r\rangle, \quad \forall r \in \mathbb{R}^n. \tag{5}$$

From the above, it is immediate that, if B is a maximal S free convex set, then the function ψ_B is a minimal valid function.

The main use of Theorem 1 is in the proof of the converse statement, namely, that every minimal valid function is of the form ψ_B for some maximal S-free convex set B containing f in its interior. The proof outline is as follows. Suppose ψ is a minimal valid function. Thus ψ is sublinear and B_ψ is an S-free convex set. Let $K := \{x - f \,|\, x \in B_\psi\}$.

i) Since $\{x \in \mathbb{R}^n \mid \rho_K(x-f) \leq 1\} = B_\psi$, Theorem 1 implies that $\psi \geq \rho_K$.

ii) Theorem [2]. *There exists a maximal S-free convex set $B = \{x \in \mathbb{R}^n : \langle a_i, x-f \rangle \leq 1,\ i = 1, \ldots, k\}$ such that $a_i \in \overline{\mathrm{conv}}(\hat{K})$ for $i = 1, \ldots, k$.*

iii) For every $r \in \mathbb{R}^n$, we have $\psi(r) \geq \rho_K(r) = \sup_{y \in \overline{\mathrm{conv}}(\hat{K})} \langle y, r \rangle \geq \max_{i=1,\ldots,k} \langle a_i, r \rangle = \psi_B(r)$, where the first inequality follows from i) and the second from ii). Since ψ_B is a valid function, it follows by the minimality of ψ that $\psi = \psi_B$.

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